GCSE Mathematics Higher Tier

Number · Algebra · Ratio, proportion and rates of change · Geometry & measures · Probability · Statistics

Here is pretty much all the Higher Tier content we could fit onto an A3 sheet of paper, including all the formulae you are required to know for GCSE. An ➔ points to an illustrative example. The codes refer to the DfE subject content. Pin this to a wall, keep it on your desk, carry it in your bag, make notes on it (sorry, don't take it into the examination)...

Listing strategies — N5

Product rule for counting:
➔ **$4 \times 3 \times 2 \times 1 = 24$ ways to arrange the letters P, I, X and L.**

Powers and roots — N6, N7

Special indices: for any value a:

$$a^0 = 1$$
$$a^{-n} = \frac{1}{a^n}$$
$$a^{\left(\frac{p}{q}\right)} = \sqrt[q]{a^p}$$

➔ $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

➔ $8^{\left(\frac{2}{3}\right)} = \sqrt[3]{8^2} = 4$

Surds — N8

Look for the biggest square number factor of the number:
➔ $\sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$

Rationalise the denominator — N8

Multiply the numerator and denominator by an expression that makes the denominator an integer:

➔ $\frac{4}{\sqrt{7}} = \frac{4 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{4\sqrt{7}}{7}$

➔ $\frac{2}{4 + \sqrt{5}}$

$= \frac{2}{4 - \sqrt{5}} \times \frac{4 - \sqrt{5}}{4 - \sqrt{5}} = \frac{2(4 - \sqrt{5})}{11}$

Standard form — N9

Standard form numbers are of the form $a \times 10^n$, where $1 \le a < 10$ and n is an integer.

Recurring decimals — N10

Make a recurring decimal a fraction:
➔ $n = 0.2\dot{3}\dot{6}$
(two digits are in the recurring pattern, so multiply by 100)
$100n = 23.\dot{6}$
(this is the same as $23.6\dot{3}\dot{6}$)
$99n = 23.6\dot{3}\dot{6} - 0.2\dot{3}\dot{6} = 23.4$
$n = \frac{23.4}{99} = \frac{234}{990} = \frac{13}{55}$

Error intervals — N15

Find the range of numbers that will round to a given value:
➔ **$x = 5.83$ (2 decimal places)**
$5.825 \le x < 5.835$
➔ **$y = 46$ (2 significant figures)**
$45.5 \le y < 46.5$
Note use of $\le$ and $<$, and that the last significant figure of each is 5.

Equations and identities — A3

An equation is true for some particular value of x...
➔ **$2x + 1 = 7$ is true if $x = 3$**
...but an identity is true for every value of x
➔ **$(x + a)^2 \equiv x^2 + 2ax + a^2$**
(note the use of the symbol $\equiv$)

Laws of indices — A4

For any value a:

$$a^x \times a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$(a^x)^y = a^{xy}$$

➔ $\left(\frac{2pq^4}{p^3q}\right)^3 = \frac{8p^3q^{12}}{p^9q^3} = \frac{8q^9}{p^6}$ or $8q^9p^{-6}$

Difference of two squares — A4

$$a^2 - b^2 = (a + b)(a - b)$$

➔ **$x^2 - 25 = (x + 5)(x - 5)$**

Rearranging a formula — A5

The subject of a formula is the term on its own. Rearrange to
➔ **Make x the subject of**
$2x + ay = y - bx$
$2x + bx = y - ay$
$x(2 + b) = y - ay$
$x = \frac{y - ay}{2 + b}$

Functions — A7

Combining functions:
$fg(x) = f(g(x))$
➔ **If $f(x) = x + 3$ and $g(x) = x^2$**
$fg(x) = x^2 + 3$
$gf(x) = (x + 3)^2$

The inverse of f is f^{-1}
➔ **If $f(x) = 2x + 5$ then**
$f^{-1}(x) = \frac{x - 5}{2}$

y = mx + c — A9

Equation of straight line $y = mx + c$
m is the gradient; c is the y intercept:
➔ **Find the equation of the line that joins $(0, 3)$ to $(2, 11)$**
Find its gradient...
$\frac{11 - 3}{2 - 0} = \frac{8}{2} = 4$
...and its y intercept...
Passes through $(0, 3)$, so c = 3.
Equation is $y = 4x + 3$.

Parallel lines: gradients are equal; perpendicular lines: gradients are "negative reciprocals".
➔ **$y = 2x + 3$ and $y = 2x - 5$ are parallel to each other; $y = 2x + 3$ and $y = -\frac{1}{2}x + 3$ are perpendicular**

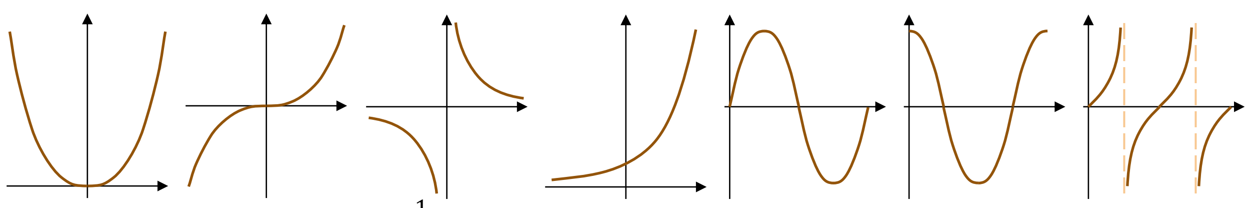
Transformations of curves — A13

Starting with the curve $y = f(x)$:
Translate $\binom{0}{a}$ for $y = f(x) + a$
Translate $\binom{-a}{0}$ for $y = f(x + a)$
Reflect in x axis for $y = -f(x)$
Reflect y axis for $y = f(-x)$

Velocity - time graph — A15

Gradient = acceleration (you may need to draw a tangent to the curve at a point to find the gradient);
Area under curve = distance travelled.

Standard graphs — A12



$y = x^2$ $y = x^3$ $y = \frac{1}{x}$ $y = a^x$ $y = \sin(x°)$ $y = \cos(x°)$ $y = \tan(x°)$

Quadratics — A11, A18

If a quadratic equation cannot be factorised, use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

➔ **Solve $2x^2 + 3x - 7 = 0$**

$x = \frac{-3 - \sqrt{9 - (-56)}}{2 \times 2} = -2.73$

or $x = \frac{-3 + \sqrt{9 - (-56)}}{2 \times 2} = 1.23$

Complete the square to find the turning point of a quadratic graph.
➔ $y = x^2 - 6x + 2$
$y = (x - 3)^2 - 9 + 2$
$y = (x - 3)^2 - 7$
Turning point is at $(3, -7)$

Equation of a circle — A16

$x^2 + y^2 = r^2$ is a circle with centre $(0, 0)$ and radius r.
➔ **$x^2 + y^2 = 25$ has centre $(0, 0)$ and radius 5.**

Simultaneous equations — A19

One linear, one quadratic;
➔ **Solve** $\begin{cases} x + 3y = 10 \\ x^2 + y^2 = 20 \end{cases}$

Rearrange the linear, and substitute into the quadratic
$x = 10 - 3y$
so $(10 - 3y)^2 + y^2 = 20$
Expand and solve the quadratic
$100 - 60y + 9y^2 + y^2 = 20$
$10y^2 - 60y + 80 = 0$
$y = 2$ or $y = 4$
Finally, substitute into the linear and solve, pairing values...
$x + 3 \times 2 = 10$ so $(x, y) = (4, 2)$
$x + 3 \times 4 = 10$ so $(x, y) = (-2, 4)$

Sequences — A24, A25

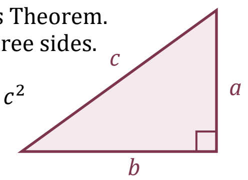
nth term of an arithmetic (linear) sequence is $bn + c$
➔ **nth term of 5,8,11,14,... is 3n+2 (always increases by 3; first term is $3 \times 1 + 2 = 5$.)**
nth term of a quadratic sequence is $an^2 + bn + c$
➔ **First three terms of $n^2 + 3n - 1$ are 3, 9, 17, ...**
Geometric sequence; multiply each term by a constant ratio
➔ **3, 6, 12, 24, ... (ratio is 2)**
Fibonacci sequence; make the next term by adding the previous two ...
➔ **2, 4, 6, 10, 16, 26, 42, ...**

Iteration — A20

You will be given the formula to use:
➔ **Solve $x^3 + 6x + 4 = 0$ by using the iteration $x_{n+1} = \sqrt[3]{6x_n - 4}$.**
Start with $x_1 = -2.8$.
$x_2 = \sqrt[3]{6 \times (-2.8) - 4} = -2.750\ldots$
$x_3 = \sqrt[3]{6 \times (-2.750\ldots) - 4} = \cdots$
Repeat until you know the solution, or you do as many as the question says.

Right angled triangles — G20

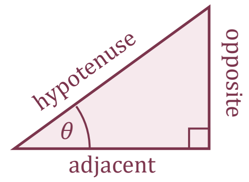
Pythagoras Theorem.
Links all three sides.
No angles.
$a^2 + b^2 = c^2$



Trigonometry.
Links two sides and one angle.
SOH | CAH | TOA

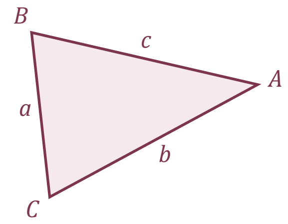
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$

Use "2ndF" or "SHIFT" key to find a missing angle



The longest side of any right angled triangle is the hypotenuse; check that your answer is consistent with this.

Advanced trigonometry — G21, G22



A is opposite *a*
B is opposite *b*
C is opposite *c*

Sine Rule
Use if you are given an angle-side pair

Missing side: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Missing angle: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Rule
Use if you can't use the sine rule

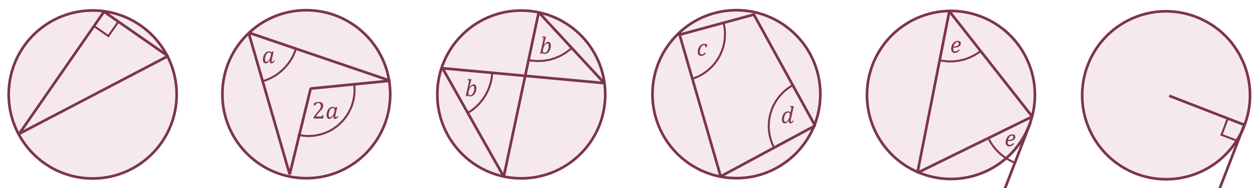
Missing side: $a^2 = b^2 + c^2 - 2bc\cos A$

Missing angle: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Special values of sin, cos, tan
Learn (or be able to find without a calculator)...

$\sin 0° = 0$,	$\cos 0° = 1$,	$\tan 0° = 0$
$\sin 30° = \frac{1}{2}$,	$\cos 30° = \frac{\sqrt{3}}{2}$,	$\tan 30° = \frac{1}{\sqrt{3}}$
$\sin 45° = \frac{1}{\sqrt{2}}$,	$\cos 45° = \frac{1}{\sqrt{2}}$,	$\tan 45° = 1$
$\sin 60° = \frac{\sqrt{3}}{2}$,	$\cos 60° = \frac{1}{2}$,	$\tan 60° = \sqrt{3}$
$\sin 90° = 1$,	$\cos 90° = 0$	

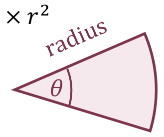
Circle theorems — G10



Angle in a semicircle is 90° · Angle at the centre is double the angle at the circumference · Angles in the same segment are equal · Opposite angles in a cyclic quadrilateral total 180° · Alternate segment theorem · Tangent and radius are perpendicular

Areas and volumes — G16, G17, G18, G23

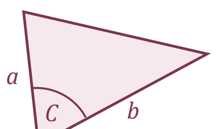
Circumference of circle = $\pi \times D$
Area of circle = $\pi \times r^2$

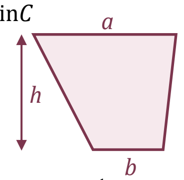


Arc length = $\frac{\theta}{360°} \times \pi \times D$

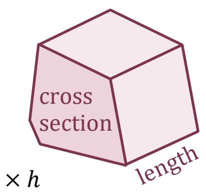
Area of sector = $\frac{\theta}{360°} \times \pi \times r^2$

Area of triangle = $\frac{1}{2}ab\sin C$

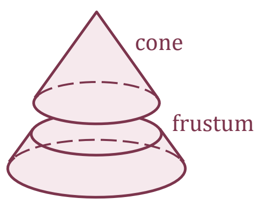




Area of trapezium = $\frac{1}{2}(a + b) \times h$



Volume of prism = area of cross section × length



Volume of cone = $\frac{1}{3}\pi r^2 h$
Volume of frustum is difference between the volumes of two cones

Transformations — G7, G8

Reflection
• Line of reflection
Translation
• Vector

Rotation
• Centre of rotation
• Angle of rotation
• Clockwise or anticlockwise

Enlargement
• Centre of enlargement
• Scale factor (if $-1 <$ SF < 1 the shape will get smaller).

Similar shapes — G19

Ratios in similar shapes and solids:

• Length/perimeter	$1 : n$	$a : b$
• Area	$1 : n^2$	$a^2 : b^2$
• Volume	$1 : n^3$	$a^3 : b^3$

Percentages: multipliers — R9, R16

Percentage increase or decrease; use a multiplier (powers for repetition)
➔ **Initially there were 20 000 fish in a lake. The number decreases by 15% each year. Estimate the number of fish after 6 years.**
$20\,000 \times 0.85^6 = 7\,500$ (2sf)

Formula for compound interest

$$\text{Total accrued} = P\left(1 + \frac{r}{100}\right)^n$$

➔ **I invest £600 at 3% compound interest. What is my account worth after 5 years?**
$£600 \times \left(1 + \frac{3}{100}\right)^5 = £695.56$

Direct & inverse proportion — R10

y is directly proportional to x:
$y = kx$ for a constant k
➔ ***b* is directly proportional to a^2; $a = 6$ when $b = 90$. Find b if $a = 8$.**
$b = ka^2$; $a = 6$ and $b = 90$ for k;
$90 = k \times 6^2$ so $k = 2.5$, $b = 2.5a^2$
$b = 2.5 \times 8^2 = 160$
y is inversely proportional to x:
$yx = k$ or $y = \frac{k}{x}$ for a constant k

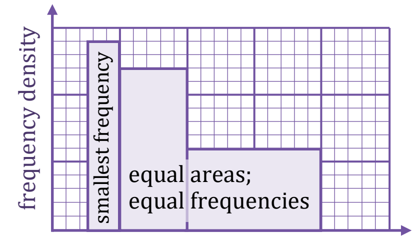
Probability rules — P8, P9

Multiply for independent events
➔ **P(6 on dice and H on coin)**
$\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$
Add for mutually exclusive events
➔ **P(5 or 6 on dice)**
$\frac{1}{6} + \frac{1}{6} = \frac{2}{6}$
Apply these rules to tree diagrams.

In general...
P(A or B) = P(A) + P(B) − P(A and B)
P(A and B) = P(A given B) × P(B)

Histograms — S3

Frequency = frequency density multiplied by class width. This means that bars with the same frequency have the same area.



Box plots — S4

Interquartile range (IQR) = UQ − LQ